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# THE ADDITIVE SUBGROUP GENERATED BY A POLYNOMIAL

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ABSTRACT
Suppose $R$ is a prime ring with the center $Z$ and the extended centroid $C$. Let $p(x_1, \ldots, x_n)$ be a polynomial over $C$ in noncommuting variables $x_1, \ldots, x_n$. Let $I$ be a nonzero ideal of $R$ and $A$ be the additive subgroup of $RC$ generated by $\{p(a_1, \ldots, a_n) : a_1, \ldots, a_n \in I\}$. Then either $p(x_1, \ldots, x_n)$ is central valued or $A$ contains a noncentral Lie ideal of $R$ except in the only one case where $R$ is the ring of all $2 \times 2$ matrices over GF(2), the integers mod 2.

In what follows, $R$ will always be an associative prime ring. $Z(R)$ will stand for the center of $R$ and $C$, for its extended centroid. For $x, y \in R$, $[x, y] = xy - yx$. For subsets $S$, $T$ of $R$, $[S, T]$ denotes the additive subgroup generated by $[s, t]$, $s \in S$, $t \in T$. By a Lie ideal of $R$, we mean an additive subgroup $U$ of $R$ such that $[U, R] \subseteq U$. As in [3], we call a Lie ideal $U$ *proper* if $[M, R] \subseteq U$ for some nonzero ideal $M$ of $R$.

Let $p(x_1, \ldots, x_n)$ be a polynomial over $C$ in noncommuting variables $x_1, \ldots, x_n$. Our main objective is the following

THEOREM. *Let $I$ be a nonzero ideal of $R$ and $A$ be the additive subgroup (of $RC$) generated by $\{p(a_1, \ldots, a_n) : a_1, \ldots, a_n \in I\}$. Then either $p(x_1, \ldots, x_n)$ is central valued or $A$ contains a proper Lie ideal of $R$, except in the only one case where $R$ is the ring of all $2 \times 2$ matrices over* GF(2), *the integers* mod 2.

In [9], some special cases of our main theorem are proved. But there $p$ is assumed to be *strongly noncentral*. Actually, the most intricate part of our proof is exactly devoted to removing this assumption. Also in [9], $R$ is assumed to be a *k-algebra*, where $k$ is a commutative ring with 1, and it is the *k-submodule* generated by all specialization of $p$, the so-called extended range of $p$, that is shown to contain noncentral Lie ideals. In this context, our theorem

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gives the most general possible formulation. These remarks may justify the length of this paper.

It is proved in [2] and [5] that in a prime ring $R$, any noncentral Lie ideal $U$ of $R$ is proper except only when $\mathrm{Ch}\, R = 2$ and $R$ is 4-dimensional over its center. Conversely, for any nonzero ideal $I$ of $R$, $[I, I]$, the additive subgroup generated by $[x, y]$, $x, y \in I$, can be seen easily to be a Lie ideal. As $p(x, y) = [x, y]$ is simply a special instance of polynomials in noncommuting variables $x$, $y$, our theorem can be viewed as a generalization of this fact.

Our theorem is very useful in reducing *linear* problems about polynomials to those about Lie ideals. Before proceeding to the proof, we give an application here. Let $\delta$, $d$ be two *nonzero* derivations on a prime ring $R$. Posner [11] and Herstein [6] proved that if $\delta d(R) \subseteq Z$, then $\mathrm{Ch}\, R = 2$, $d^2 = 0$ and $\delta = \lambda d$ for some $\lambda \in C$ except only when $\mathrm{Ch}\, R = 2$ and $R$ is 4-dimensional over its center. Lee [10] and Ke [8] generalizes the above result by assuming only $\delta d(U) \subseteq Z$, where $U$ is a noncentral Lie ideal. Our theorem gives immediately the following further generalization.

COROLLARY. *Let $\delta$, $d$ and $R$ be as explained above. Let $I$ be a nonzero ideal of $R$ and $p(x_1, \ldots, x_n)$ be a noncentral polynomial with coefficients over $C$. If $\delta d(p(a_1, \ldots, a_n) \in Z$ for $a_1, \ldots, a_n \in I$, then* $\mathrm{Ch}\, R = 2$, $d^2 = 0$ *and* $\delta = \lambda d$ *for some* $\lambda \in C$.

The corollary above is also a generalization of the following result due to Felzenszwalb [4]: Let $d$ be a *nonzero* derivation on a prime ring $R$. Let $n \geqq 1$ be a fixed integer. Suppose $d(x^n) = 0$ for all $x \in R$. Then $R$ is commutative.

Now we come to the proof of our main theorem, which is divided into three cases. We begin with the easiest.

## 1. The case of non-p.i. rings

Suppose $R$ is not a p.i. ring. By linearizing, we may assume $p(x_1, \ldots, x_n)$ is multilinear. Replacing $I$ by a smaller one, we may assume $p(a_1, \ldots, a_n) \in R$ for $a_1, \ldots, a_n \in I$ and hence $A \subseteq R$. Let $d$ be an inner derivation of $R$. Observe that $d(I) \subseteq I$. Using the multilinearity of $p(x_1, \ldots, x_n)$, we have

$$d(p(a_1, \ldots, a_n)) = p(d(a_1), a_2, \ldots, a_n) + p(a_1, d(a_2), \ldots, a_n) + \cdots + p(a_1, \ldots, d(a_n)) \in A.$$

Hence $[R, A] \subseteq A$ and $A$ itself is a Lie ideal. $A$ must be proper, for otherwise $R$ would be a p.i. ring by lemma 1 [2] and theorem 1.5 [5].

## 2. The case of p.i. rings with finite center

Let $R$ be a p.i. ring with $Z = Z(R)$ finite. By theorem 2 (p. 57 [7]) $R = R_z = M_s(Z) =$ the ring of all $s \times s$ matrices over $Z$, where $s^2 = \dim_z R$. $R$ is simple and so $I = R$. If $s = 1$, then $R$ is commutative and $p(x_1, \ldots, x_n)$ is trivially central valued. If $s > 1$, then $R$ possesses a nontrivial idempotent. Observe that $A$ is invariant under all automorphisms of $R$. Our desired result follows immediately from the theorems of [3].

## 3. The case of p.i. rings with infinite center

This is the stickiest part of the proof. Suppose $R$ is a p.i. ring with $Z(R)$ infinite. If $R$ is not a domain, then we can also argue as in the previous case. But our proof given below works for both domains and nondomains.

Let $\dim_z R = s^2$ and $K =$ the quotient field of $Z(R)$. Note that $K$ coincides with the extended centroid $C$. Following the notations in [7], set $K\{X\} = K\{x_1, \ldots\} =$ the polynomial ring over $K$ in noncommuting variables $x_1, x_2, \ldots$ . We remark that $x, y, z, t, \ldots$ will also be used to denote $x_1, x_2, \ldots$ for convenience. Set $K\{\bar{X}\} = K\{X\}/I_s$, where $I_s$ is the $T$-ideal of identities of $M_s(K)$. Note that $K\{\bar{X}\}$ is a domain (theorem 2, p. 90, [7]) and its ring of central quotients, denoted by $UD(K, s)$, is a division algebra of degree $s$. Let $Z\{X\}$ denote the subring of $K\{X\}$ consisting of polynomials with coefficients in $Z = Z(R)$ and $Z\{\bar{X}\}$ be the homomorphic image of $Z\{X\}$ under the natural homomorphism from $K\{X\}$ onto $K\{\bar{X}\}$ (sending $x_i$ to $\bar{x}_i$).

We recall some more definitions. An element $f \in K\{X\}$ is said to be homogeneous in $x_i$ if each term (monomial) of $f$ has the same $x_i$-degree. If $f$ is homogeneous in each variable, then $f$ is said to be completely homogeneous. We say $f$ is of degree $\langle k_1, k_2, \ldots, k_m \rangle$ if the $x_i$-degree of $f$ is $k_i$ for $i = 1, \ldots, m$ and is 0 for $i > m$. It is obvious that every $f \in K\{X\}$ can be written uniquely as a sum of completely homogeneous polynomials of *distinct* degrees. We write $f = \Sigma f_{\langle k_1, k_2, \ldots, k_m \rangle}$, where $f_{\langle k_1, k_2, \ldots, k_m \rangle}$ denotes the completely homogeneous part of $f$ of degree $\langle k_1, \ldots, k_m \rangle$. Finally we define

$$A_0 = \text{the additive subgroup of } K\{\bar{X}\} \text{ generated by } p(s_1, \ldots, s_n),\ s_1, \ldots, s_n \in Z\{\bar{X}\}.$$

$$A_1 = \{f(x_1, \ldots, x_m) \in K\{X\} : f(\bar{x}_1, \ldots, \bar{x}_m) \in A_0\}.$$

Observe that, for $f(x_1, \ldots, x_m) \in A_1$ and $a_1, \ldots, a_m \in I$, we have $f(a_1, \ldots, a_m) \in A$.

LEMMA 1. *Assume $A_1$ contains an element $f(x_1, \ldots, x_m)$ such that*
(1) *$f$ is not central and*
(2) *$f$ is linear in some $x_i$ $(i = 1, \ldots, m)$, say in $x_1$.*
*Then our theorem holds.*

PROOF. Choose $f \in A_1$ which satisfies, in addition to (1) and (2) above, also the following
(3) *$f$ has minimum number of completely homogeneous parts with respect to (1) and (2).*

We claim $f$ is completely homogeneous. Suppose $f$ involves $x_1, \ldots, x_m$ only. Since $f$ is not central, one of its completely homogeneous part, say $f_{\langle 1, k_2, \ldots, k_m \rangle}$, is noncentral. It suffices to show $f$ has no other completely homogeneous parts than $f_{\langle 1, k_2, \ldots, k_m \rangle}$. Suppose, on the contrary, that $f_{\langle 1, l_2, \ldots, l_m \rangle}$ is another one distinct from $f_{\langle 1, k_2, \ldots, k_m \rangle}$. For $\alpha \in Z(R)$,

$$f(x_1, \alpha x_2, \ldots) = \alpha^{k_2} f_{\langle 1, k_2, \ldots \rangle}(x_1, \ldots) + \alpha^{l_2} f_{\langle 1, l_2, \ldots \rangle}(x_1, \ldots) + \cdots,$$

and, using the $x_1$-linearity

$$\begin{aligned} f(\alpha^{l_2} x_1, x_2, \ldots) &= \alpha^{l_2} f(x_1, x_2, \ldots) \\ &= \alpha^{l_2} f_{\langle 1, k_2, \ldots \rangle}(x_1, \ldots) + \alpha^{l_2} f_{\langle 1, l_2, \ldots \rangle}(x_1, \ldots) + \cdots. \end{aligned}$$

(Dots above denote summation over completely homogeneous parts other than $f_{\langle 1, k_2, \ldots \rangle}$ and $f_{\langle 1, l_2, \ldots \rangle}$.) So we have

$$\begin{aligned} g(x_1, \ldots) &= f(x_1, \alpha x_2, \ldots) - f(\alpha^{l_2} x_1, x_2, \ldots) \\ &= (\alpha^{k_2} - \alpha^{l_2}) f_{\langle 1, k_2, \ldots \rangle} + \cdots. \end{aligned}$$

Obviously $g \in A_1$ and $g$ contains less completely homogeneous parts than $f$ since $f_{\langle 1, l_2, \ldots \rangle}$ has been canceled. By (3) above, $g$ is central and so is its completely homogeneous part $(\alpha^{k_2} - \alpha^{l_2}) f_{\langle 1, k_2, \ldots \rangle}$. But $f_{\langle 1, k_2, \ldots \rangle}$ has been assumed to be noncentral. We have $\alpha^{k_2} = \alpha^{l_2} = 0$. This holds for all $\alpha$ lying in the infinite set $Z(R)$. $k_2 = l_2$ follows. Using the same argument, we can show $k_3 = l_3, \ldots, k_m = l_m$. This is absurd!

By replacing our $p$ by $f$ above, we may assume our $p$ is linear in $x_1$ and is completely homogeneous. Set $Z_0 =$ the center of $Z\{\bar{X}\}$ and $Z_2 =$ the center of $UD(K, s)$ Define

$$B = \{ g(\bar{x}_1, \ldots.)\alpha^{-1} : g(\bar{x}_1, \ldots) \in A_0 \text{ and } \alpha \in Z_0 - \{0\} \}.$$

Since $p$ is linear in $x_1$ and is completely homogeneous, $B$ is equal to the additive subgroup of $UD(K, s)$ generated by $p(d_1, \ldots, d_n)$, $d_1, \ldots, d_n \in UD(K, s)$. So $B$ is an invariant $Z_2$-subspace of $UD(K, s)$. By theorem 7 [1], $B \supseteq [UD(K, s), UD(K, s)]$. Thus there exists a nonzero central polynomial $c(x, y)$ in $Z\{X\}$ such that $c(x, y)[x, y] \in A_1$. Note that we have left out variables other than $x, y$ in $c(x, y)$. Since $p$ is linear in $x_1$, $A_1$ is a $Z(R)$-module. Using the fact that $Z(R)$ is infinite, we may assume $c(x, y)$ is completely homogeneous. Let $z$ be a new variable. Linearizing $x$, we have

$$\sigma = c(x+z, y)[x+z, y] - c(x, y)[x, y] - c(z, y)[z, y] \in A_1.$$

Set $l$ = the $x$-degree of $c(x, y)$ and let $\tau$ = the sum of terms in $\sigma$ whose $z$-degee is $l$. Rewrite $\sigma$ as

$$\sigma = (c(x+z, y) - c(x, y))[x, y] + (c(x+z, y) - c(z, y))[z, y];$$

we can see that $\tau$ is of the form

$$\tau = c(z, y)[x, y] + c'(x, z, y)[z, y]$$

where $c'(x, z, y)$ is another central element in $Z\{X\}$. Using the fact that $A_1$ is a $Z(R)$-module and $Z(R)$ is infinite, we can see that there exists $\alpha \in Z(R) - \{0\}$ such that $\alpha\tau \in A_1$. Since $p$ is linear in $x_1$, $A_1$ is also closed under multiplication by central elements of $Z\{X\}$. So $\alpha c(z, y)\tau \in A_1$. Write

$$\alpha c(z, y)\tau = \alpha c(z, y)^2[x, y] + \alpha c'(x, z, y)c(z, y)[z, y] \in A_1$$

and note

$$\alpha c'(x, z, y)c(z, y)[z, y] \in \alpha c'(x, z, y)A_1 \subseteq A_1.$$

We have $\alpha c(z, y)^2[x, y] \in A_1$. Repeating this argument for $y$, we have $\beta c(z, w)^4[x, y] \in A_1$ for some $\beta \in Z(R) - \{0\}$ and another new variable, $w$. Now pick elements of $I$ so that the evaluation of $\beta c(z, w)^4$ on these elements is some nonzero $\gamma \in Z$. Then $A \supseteq \gamma[I, I]$ as desired.

To produce $f \in A_1$ as described in Lemma 1, we need some sort of linearization. Let us recall the *difference operator* $\Delta_y^{x_i}$ in $K\{X\}$ (p. 16, [7]): Let $f(x_1, \ldots, x_m) \in K\{X\}$. Then for $1 \leqq i \leqq m$, we define

$$\Delta_y^{x_i} f(x_1, \ldots, x_m) = f(x_1, \ldots, x_{i-1}, x_i + y, x_{i+1}, \ldots, x_m) - f(x_1, \ldots, x_i, \ldots, x_m) - f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_m).$$

We also need the *differential operator* $D_y^{x_i}$ which we define right here.

(1) Let $m(\ldots, x_i, \ldots)$ be a monomial in $K\{X\}$. A monomial $g$ is called *a*

*variation of $m$ at $x_i$* by $y$ if $g$ is obtained from $m$ by substituting $y$ for *one* occurrence of $x_i$ in $m$. In case that $m$ does not involve $x_i$, we set *the* variation of $m$ at $x_i$ by $y$ to be 0.

E.g. Let $m(x_1, x_2) = x_1^2 x_2 x_1$. Then $yx_1x_2x_1$, $x_1yx_2x_1$, $x_1^2x_2y$ are (all the) variations of $m$ at $x_i$ by $y$.

(2) For a monomial $m$ in $K\{X\}$, $D_y^{x_i}m$ is defined to be the sum of *all* distinct variations of $m$ at $x_i$ by $y$.

E.g. For $m$ in the example above, we have $D_y^{x_1}m = yx_1x_2x_1 + x_1yx_2x_1 + x_1^2x_2y$.

(3) For $f \in K\{X\}$ in general, write $f = \Sigma m_k$ where $m_k$'s are monomials. Define $D_y^{x_i}f$ to be $\Sigma D_y^{x_i}m_k$. The well-definedness is obvious.

The following lemma is what we need about these two operators.

LEMMA 2. *Let $f(x_1, \ldots, x_m) \in K\{X\}$ involve only $x_1, \ldots, x_m$.*

(1) *Suppose $d$ is an inner derivation of $R$. Then for $a_1, \ldots, a_m \in R$,*

$$df(a_1, \ldots, a_m) = \sum_{i=1}^{m} D_{d(a_i)}^{a_i} f(a_1, \ldots, a_m).$$

*(Here we abuse the notations in an obvious way.)*

(2) *Suppose the $x_i$-degree of each monomial of $f$ is larger than 1. Then $D_y^{x_i}f =$ the sum of terms in $\Delta_y^{x_i}f$ whose $y$-degree is 1.*

(3) *Write $f = \Sigma f_{\langle k_1, \ldots, k_m \rangle}$, where $f_{\langle k_1, \ldots, k_m \rangle}$ is the completely homogeneous part of $f$ of degree $\langle k_1, \ldots, k_m \rangle$. If $\Delta_y^{x_i}f$ is central for all $i = 1, \ldots, m$, then each completely homogeneous part $f_{\langle k_1, \ldots, k_m \rangle}$ such that $k_1 > 1, \ldots, k_m > 1$ must be central.*

PROOF. (1) and (2): It is obvious by direct expansion when $f$ is a monomial. Since $d$, $D_y^{x_i}$, and $\Delta_y^{x_i}$ are all linear, the result extends easily to arbitrary $f \in K\{X\}$.

(3) Suppose $k_1 > 1, \ldots, k_m > 1$. By (1) and (2) above, $df_{\langle k_1, \ldots, k_m \rangle}(a_1, \ldots, a_m)$ is central for any inner derivation $d$. This implies easily that $f_{\langle k_1, \ldots, k_m \rangle}$ is central.

Let us choose $f \in A_1$ such that

(1) $f$ is noncentral.

(2) $f$ is of minimum height with respect to (1). (Height is defined in p. 15 [7].)

(3) $f$ contains minimum number of distinct nonzero completely homogeneous parts with respect to (1) and (2).

LEMMA 3. *Suppose* $f = f(x_1, \ldots, x_m) \in A_1$ *satisfies* (1), (2), (3) *above. Then* (1) *there is only one completely homogeneous part of $f$ that is noncentral and* (2) *for this noncentral completely homogeneous part, say* $f_{\langle k_1, \ldots, k_m \rangle}$, *one of* $k_1, \ldots, k_m$ *must be* 1.

PROOF. By (3) above, $f$ must be blended. We may assume $k_1 \geqq 1, \ldots, k_m \geqq 1$ for each completely homogeneous part $f_{\langle k_1, \ldots, k_m \rangle}$. Since $f$ is assumed to be noncentral, one of its completely homogeneous parts, say $f_{\langle k_1, \ldots, k_m \rangle}$, is noncentral. Suppose $f_{\langle l_1, \ldots, l_m \rangle}$ is another completely homogeneous part distinct from $f_{\langle k_1, \ldots, k_m \rangle}$. Then for $\alpha \in Z(R)$,

$$f(\alpha^{l_2} x_1, x_2, \ldots) = \alpha^{k_1 l_2} f_{\langle k_1, \ldots, k_m \rangle}(x_1, \ldots, x_m) + \alpha^{l_1 l_2} f_{\langle l_1, \ldots, l_m \rangle}(x_1, \ldots, x_m) + \cdots,$$

$$f(x_1, \alpha^{l_1} x_2, \ldots) = \alpha^{k_2 l_1} f_{\langle k_1, \ldots, k_m \rangle}(x_1, \ldots, x_m) + \alpha^{l_1 l_2} f_{\langle l_1, \ldots, l_m \rangle}(x_1, \ldots, x_m) + \cdots.$$

Let $g(x_1, x_2, \ldots) = f(\alpha^{l_2} x_1, x_2, \ldots) - f(x_1, \alpha^{l_1} x_2, \ldots)$. Then $g$ takes the form

$$g(x_1, \ldots, x_m) = (\alpha^{k_1 l_2} - \alpha^{k_2 l_1}) f_{\langle k_1, \ldots, k_m \rangle}(x_1, \ldots, x_m) + \cdots;$$

$g$ contains less completely homogeneous parts than $f$. It is also obvious that $g \in A_1$ and height $g \leqq$ height $f$. Hence $g$ must be central and so must its completely homogeneous part $(\alpha^{k_1 l_2} - \alpha^{k_2 l_1}) f_{\langle k_1, \ldots, k_m \rangle}$. Thus $\alpha^{k_1 l_2} = \alpha^{k_2 l_1}$ and this implies $k_1 / l_1 = k_2 / l_2$. Repeating the same argument, we have $k_1 / l_1 = k_2 / l_2 = \cdots = k_m / l_m$.

Since $f$ is of minimum height among noncentral elements of $A_1$, $\Delta_y^{x_i} f$ must be central for $i = 1, \ldots, m$. By (3) of Lemma 2, if $f_{\langle i_1, \ldots, i_m \rangle}$ is noncentral, then one of $i_1, \ldots, i_m$ must be 1. Hence one of $k_1, \ldots, k_m$, say $k_1$, is 1. For any $f_{\langle l_1, \ldots, l_m \rangle}$ other than $f_{\langle k_1, \ldots, k_m \rangle}$, we have $l_2 = k_2 l_1, \ldots, l_m = k_m l_1$ by the result of the previous paragraph. If $l_1 = 1$, then $l_2 = k_2, \ldots, l_m = k_m$. This contradicts the distinctness of $f_{\langle 1, k_2, \ldots, k_m \rangle}$ and $f_{\langle l_1, l_2, \ldots, l_m \rangle}$. Hence none of $l_1, \ldots, l_m$ can be 1. By (3) of Lemma 2 again $f_{\langle l_1, \ldots, l_m \rangle}$ must be central. This finishes the proof.

Let $f_{\langle k_1, \ldots, k_m \rangle}$ with $k_1 = 1$ be the only noncentral completely homogeneous part of $f$. Write $f_{\langle k_1, \ldots, k_m \rangle} = f_1$ for short. Then $f = c_0 + f_1$, where $c_0$ is a central polynomial. A little reflection will show this is the best form that linearization process can give. To obtain our desired $f$, we need another technique, which is contained in

LEMMA 4. $A_1$ *does contain an element as described in Lemma* 1.

PROOF. Let $t_1$ be a new variable distinct from $x_1, \dots, x_m$. By theorem 1 on p. 44 of [7], there is $h(\boldsymbol{t}) = h(t_1, t_2, \dots) \in Z\{X\}$ such that $t_1 h(\boldsymbol{t}) = h(\boldsymbol{t}) t_1 = c(\boldsymbol{t})$, where $c(\boldsymbol{t})$ is a nonzero central polynomial. We may assume the variables $t_1, t_2, \dots$ involved in $h(\boldsymbol{t})$ are all distinct from $x_1, \dots, x_m$. We may also assume $h(\boldsymbol{t})$ is completely homogeneous. Set $g(\boldsymbol{t}, \boldsymbol{x}) = f(c(\boldsymbol{t})x_1, \dots, c(\boldsymbol{t})x_m)$. We introduce the following notations for simplicity: $\bar{t}$ for $(\bar{t}_1, \bar{t}_2, \dots)$, $\bar{x}$ for $(\bar{x}_1, \dots, \bar{x}_m)$, $\bar{h}$ for $h(\bar{t}) = h(\bar{t}_1, \bar{t}_2, \dots)$ and $\bar{c}$ for $c(\bar{t}) = c(\bar{t}_1, \bar{t}_2, \dots)$. Now working in $UD(K, s)$, we have

$$\begin{aligned} \bar{t}_1 g(\bar{t}, \bar{x})\bar{h}/\bar{c} &= \bar{t}_1 f(\bar{c}\bar{x}_1, \bar{c}\bar{x}_2, \dots, \bar{c}\bar{x}_m)\bar{t}_1^{-1} \\ &= f(\bar{t}_1\bar{c}\bar{x}_1\bar{t}_1^{-1}, \dots, \bar{t}_1\bar{c}\bar{x}_m\bar{t}_1^{-1}) \\ &= f(\bar{t}_1\bar{x}_1\bar{h}, \dots, \bar{t}_{11}\bar{x}_m\bar{h}) \in A_0. \end{aligned}$$

Assume $1 + k_2 + \cdots + k_m = l$. Using the fact that $c_0$ is central, we have

$$\bar{t}_1 g(\bar{t}, \bar{x})\bar{h}/\bar{c} = c_0(\bar{c}\bar{x}_1, \dots, \bar{c}\bar{x}_m) + \bar{t}_1 f_1(\bar{x}_1, \dots, \bar{x}_m)\bar{c}^{l-1}\bar{h}.$$

Hence

$$\begin{aligned} g(\bar{x}, \bar{t}) - \bar{t}_1 g(\bar{t}, \bar{x})\bar{h}/\bar{c} &= f_1(\bar{c}\bar{x}_1, \dots, \bar{c}\bar{x}_m) - \bar{t}_1 f_1(\bar{x}_1, \dots, \bar{x}_m)\bar{c}^{l-1}\bar{h} \\ &= \bar{c}^l f_1(\bar{x}_1, \dots, \bar{x}_m) - \bar{t}_1 f_1(\bar{x}_1, \dots, \bar{x}_m)c^{l-1}\bar{h} \in A_0. \end{aligned}$$

Thus $c(\boldsymbol{t})^l f_1(\boldsymbol{x}) - t_1 f_1(\boldsymbol{x}) c(\boldsymbol{t})^{l-1} h(\boldsymbol{t}) \in A_1$ and is linear in $x_1$. So it suffices to show this element is not central.

Suppose on the contrary that $c^l f_1 - t_1 f_1 c^{l-1} h$ is central. Working in $K\{\bar{X}\}$, we have $\bar{f}_1 - \bar{t}_1 \bar{f}_1 \bar{t}_1^{-1} = \alpha \in$ the center of $Z\{\bar{X}\}$. Replacing $t_1$ by $t_1 + 1$, we have also that $\bar{f}_1 - (\bar{t}_1 + 1)\bar{f}_1(\bar{t}_1 + 1)^{-1} = \beta \in$ the center of $Z\{\bar{X}\}$. Hence $\bar{f}_1\bar{t}_1 - \bar{t}_1 f_1 = \alpha\bar{t}_1 = \beta(\bar{t}_1 + 1)$. If $\alpha \neq \beta$, then $\bar{t}_1$ would be central, absurd! So $\alpha = \beta$ and then $\beta = 0$. So $\bar{f}_1\bar{t}_1 = \bar{t}_1\bar{f}_1$. This implies $f_1$ is central, absurd again. This completes our proof.

## REFERENCES


1. S. Asano, *On invariant subspaces of division algebras*, Kodai Math. **18** (1966), 322–334.
2. J. Bergen, I. N. Herstein and J. W. Kerr, *Lie ideals and derivations of prime rings*, J. Algebra **71** (1981), 259–267.
3. C.-L. Chuang, *On invariant additive subgroups*, Isr. J. Math. **57** (1987), 116–128.
4. B. Felzenszwalb, *Derivations in prime rings*, Proc. Amer. Math. Soc. **84** (1982), 16–20.
5. I. N. Herstein, *Topics in Ring Theory*, Univ. of Chicago Press, Chicago, 1969.
6. I. N. Herstein, *A note on derivations II*, Canad. Math. Bull. **22** (1979), 509–511.
7. N. Jacobson, *P.I. Algebras, An Introduction*, Lecture Notes in Mathematics, No. 441, Springer-Verlag, Berlin/New York, 1975.

8. W.-F. Ke, *On derivations of prime rings of characteristic* 2, Chinese J. Math. **13** (1985), 273–290.

9. A. Kovacs, *On the nonexistence of hypercommuting polynomials*, Proc. Amer. Math. Soc. **66** (1977), 241–246.

10. P. H. Lee and T. K. Lee, *Lie ideals of prime rings with derivations*, Bull. Inst. Math. Acad. Sinica **11**(1) (1983), 75–80.

11. E. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.